

DIRECTED RANDOM WALK ON THE LATTICES OF GENUS TWO

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Abstract

The object of the present investigation is an ensemble of self-avoiding and directed graphs belonging to eight-branching Cayley tree (Bethe lattice) generated by the Fuchsian group of a Riemann surface of genus two and embedded in the Poincaré unit disk. We consider two-parametric lattices and calculate the multifractal scaling exponents for the moments of the graph lengths distribution as functions of these parameters. We show the results of numerical and statistical computations, where the latter are based on a random walk model.

Keywords: directed random walk; Cayley tree.

1 Introduction

A number of stochastic models is taken up to describe the different structures on underlying non-trivial geometrical carriers like random environments, percolation clusters, fractals, etc. Restricting by physical systems and phenomena, we can note turbulence [1, 2], diffusion limited aggregation [3], polymers [4, 5], the particle dynamics in a random magnetic field [6, 7] as the examples. The last decades it seems natural to describe these complicated and chaotic systems by using the multifractality concept [8, 9] giving us a spectrum of critical exponents which can be connected with the results of renormalization group computations (see Ref. [10]).

We can also find a great number of geometrical objects, chaotic at first sight, within the Lobachevsky geometry. Indeed, even limiting by dimension two, there is a possibility to construct the infinite number of hyperbolic lattices, existence of which is not allowed in a flat space [11]. This fact inspires us in the present paper to turn to Lobachevsky geometry and to demonstrate that the structures, well studied in mathematics, might pretend to describe and generate some kind of disorder, like a random walk, analytically. Similar idea to consider a directed random walk on a Cayley tree in hyperbolic space has been already exploited in Refs. [12, 13]. We also note that the models of billiards intensively use the hyperbolic Riemann surfaces as geometrical carriers [14, 15].

In our paper, we shall watch a thread of walker moving along the eight-branching and rooted Cayley tree constructed as a dual lattice to the hyperbolic octagonal lattice tiling the Poincaré disk under the action of Fuchsian group. The octagonal lattice cells, whose opposite sides are identified, correspond topologically to the two-holed torus, geometry of which is defined here by one independent module while other two complex moduli are constrained. (Note that the Riemann surfaces of genus g are characterized by $6g - 6$ real parameters.) We characterize a walk by spectrum of lengths which result from summation of hyperbolic distances between tree sites visited and root point. Introducing a partition function and the moments of distribution function, we would like to investigate an ensemble of these lengths within the multifractality concept.

Thus, the model under consideration is deterministic by construction, and partition function is actually the truncated series over a symmetry group. To calculate this kind of series, we apply

both numerical and statistical methods, where the former is needed to exhibit the basic properties of the model, which we aim to reproduce by means of the latter approach called for purely analytical description. Within the framework of statistical method, we elaborate a scheme of length spectrum computation and reduce the partition function to the Markov multiplicative chain in a random walk approximation.

2 Space and Symmetries

We will deal with a particular case of a two-dimensional Lobachevsky space (with the Gaussian curvature $K = -1$), namely, a model of the open Poincaré disk centered at the origin,

$$\mathbb{D} = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}, \quad (1)$$

endowed with the metric:

$$ds^2 = 4 \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}. \quad (2)$$

Mainly, we are interested in the symmetries (isometries) of the metric (2), which we will use for construction of non-trivial structures on \mathbb{D} . However, an application of this metric may not be limited by symmetry production only. In Appendix A, we also develop heuristic ideas how to relate the curved space to the Euclidean two-dimensional space with potential field. Such a correspondence, in our mind, can follow from Maupertuis variational principle, constituting an equivalence between particle trajectories (written down below) in these spaces.

Solution to the geodesic equation in (\mathbb{D}, ds^2) is the functions:

$$\begin{aligned} x(s) &= \frac{\cos \phi \cosh s + R \sin \phi \sinh s}{\sqrt{1 + R^2} \cosh s + R}, \\ y(s) &= \frac{R \cos \phi \sinh s - \sin \phi \cosh s}{\sqrt{1 + R^2} \cosh s + R}, \end{aligned} \quad (3)$$

which are defined in the interval $s \in (-\infty, +\infty)$ and describe an arc inside \mathbb{D} with the radius R and the center at the point $x_0 = \sqrt{1 + R^2} \cos \phi$, $y_0 = \sqrt{1 + R^2} \sin \phi$, lying beyond the unit disk.

Now let us introduce a structure on the Poincaré disk. Here we aim to consider an example of lattice resulting from operation of a particular Fuchsian group Γ (stating “periodic law”) on the (complexified) Poincaré disk \mathbb{D} [11]. The associated quotient space \mathbb{D}/Γ is assumed to be a Riemann surface of genus $g = 2$, whose fundamental domain \mathcal{F} is hyperbolic octagon in \mathbb{D} , which can be viewed as the result of gluing two tori together.

Thus we should first construct the fundamental octagon \mathcal{F} and the corresponding Fuchsian group Γ by means of which we tile \mathbb{D} by the octagonal lattice. Note that an algorithm connecting \mathcal{F} for $g = 2$ with Fuchsian group Γ and vice versa is described in Ref. [16].

The Riemann surface of genus $g = 2$ is completely determined by a set of three complex moduli corresponding to three corners of the fundamental domain \mathcal{F} (in complex plane). The fourth corner is then found to adjust the area of the fundamental domain to be $2\pi(2g - 2) = 4\pi$, according to the Gauss-Bonnet theorem [11]. Applying inversions across the origin to these four corners yield the remaining four corners of the octagon.

For the sake of simplicity, we shall focus on symmetric octagon with the *one* arbitrary module, namely, one pair of the length and angle variables denoted as (a, α) . Such an octagon (sketched in Fig. 1) can be obtained by deformation of the regular hyperbolic octagon (with $a = 2^{-1/4}$, $\alpha = \pi/4$),

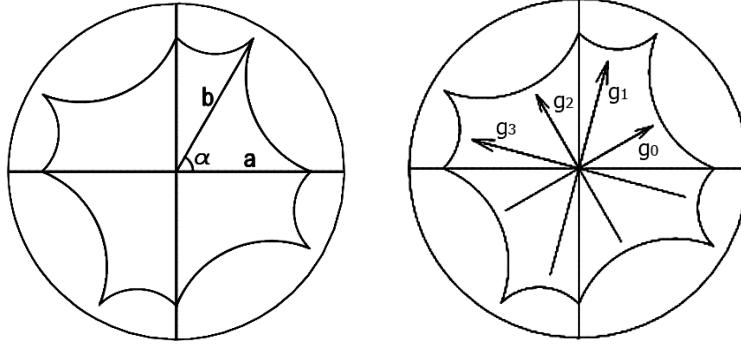


Figure 1: Symmetric octagon with $a = 0.8$, $\alpha = \pi/3$ and generators g_k of Fuchsian group.

well studied in the context of the chaology (see, for example, Refs. [14, 16] and references therein). Introducing the complex variable $z = x + iy \in \mathbb{C}$ in order to develop the further analysis in usual manner, this construction is as follows.

Let us assume that the corners of the octagon are at the points $a \exp(ik\pi/2)$, $b \exp i(\alpha + k\pi/2)$, where $0 < a, b < 1$, $k = \overline{0, 3}$. At this stage, parameter b is unknown and should be found as the function of (a, α) . Connecting the points by geodesics, we shall require the sum of inner angles (by vertices) of the octagon to be equal 2π in accordance with the Gauss-Bonnet theorem. This requirement allows us to determine b and the parameters of geodesics constituting sides of octagon. It is obvious that $b = a = 2^{-1/4}$ for the regular octagon (the fixed value of a is required because the homothety property is not valid in the space with $K \neq 0$).

In the case at a hand, the boundary of octagon, $\partial\mathcal{F}$, is formed by geodesics of two kinds (labeled by “ \pm ” below). These geodesics are completely determined by the radii R_{\pm} and the angles $\phi_{\pm} + k\pi/2$ ($k = \overline{0, 3}$), defining the positions of the circle centers. Satisfying the conditions imposed above and collected in manifest form in Appendix B, we find that

$$R_{\pm} = \frac{1}{2a} \sqrt{T_{\pm}^2 + (1 - a^2)^2}, \quad \phi_{\pm} = \arctan \left[\left(\frac{T_{\pm}}{1 + a^2} \right)^{\pm 1} \right], \quad (4)$$

where

$$T_{\pm} = a^2 \pm \tan \left(\alpha - \frac{\pi}{4} \right). \quad (5)$$

In these formulas it is supposed that $0 < \phi_+ < \alpha < \phi_- < \pi/2$. Moreover, introducing an angle β by vertices $a \exp(ik\pi/2)$ (the angle by vertices $b \exp i(\alpha + k\pi/2)$ is then equal to $\pi/2 - \beta$),

$$\begin{aligned} \tan \beta &= \frac{1 - \sqrt{1 + R_+^2} \sqrt{1 + R_-^2} \cos(\phi_- - \phi_+)}{1 - \sqrt{1 + R_+^2} \sqrt{1 + R_-^2} \sin(\phi_- - \phi_+)} \\ &= (1 - a^2) \frac{2a^2 \cos^2(\alpha - \frac{\pi}{4})}{2a^2 \cos^2(\alpha - \frac{\pi}{4}) - 1}, \end{aligned} \quad (6)$$

we should control the condition $0 < \beta < \pi/2$, defining the region of accessibility of parameters (a, α) :

$$\frac{1}{\sqrt{2} \cos(\alpha - \frac{\pi}{4})} < a < 1, \quad (7)$$

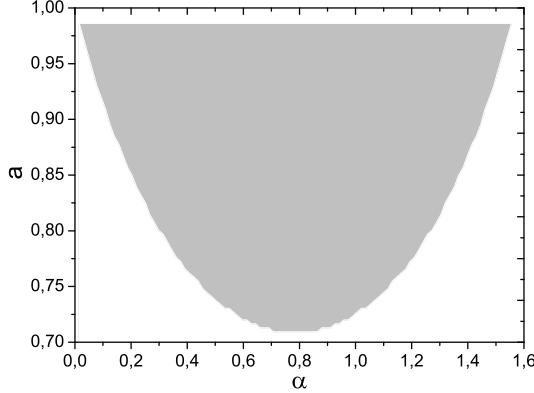


Figure 2: The permission domain of module (a, α) .

which is sketched in Fig. 2.

To complete the octagon description, the parameter b , pointed out in Fig. 1, can be calculated as

$$\begin{aligned} b &= \cos(\alpha - \phi_+) \left[\sqrt{1 + R_+^2} - \sqrt{R_+^2 - \tan^2(\alpha - \phi_+)} \right] \\ &= \frac{1}{\sqrt{2}a \cos(\alpha - \pi/4)}. \end{aligned} \quad (8)$$

Note that the manifest dependence of the octagon parameters on moduli is necessary in the different problems where geometry of the system (carrier geometry) is not fixed. For instance, (a, α) would be the (dynamical) variables of evolution parameter in topological gravity describing an universe of genus $g = 2$; it is able to average over (a, α) in statistical physics, etc.

The Fuchsian group Γ of a Riemann surface of genus two is generated by four generators g_0, g_1, g_2, g_3 and their inverses. Let generators g_k, g_k^{-1} , acting freely as isometries of Poincaré disk, map geodesic boundary segments of octagon \mathcal{F} onto each other, thereby identifying opposite edges (see Fig. 1, right panel). This definition leads to the group relation:

$$g_0 g_1^{-1} g_2 g_3^{-1} g_0^{-1} g_1 g_2^{-1} g_3 = \text{id}. \quad (9)$$

Generally, the group of orientation-preserving isometries of (\mathbb{D}, ds^2) is presented by $PSU(1, 1) = SU(1, 1)/\{\pm 1\}$ (see textbooks like Ref. [11]), where

$$SU(1, 1) = \left\{ \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix} \middle| u, v \in \mathbb{C}, |u|^2 - |v|^2 = 1 \right\}. \quad (10)$$

The group of isometries acts via fractional linear transformations:

$$z \mapsto \gamma[z] = \frac{uz + v}{\bar{v}z + \bar{u}}, \quad \gamma \in PSU(1, 1), \quad z \in \mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}. \quad (11)$$

Then the Fuchsian group Γ is a discrete group such that $\Gamma \subset PSU(1, 1)$. One also adds the hyperbolicity condition, $|\text{Tr } \gamma| > 2$ for all $\gamma \in \Gamma$. Constructing the corresponding Fuchsian group for a given octagon, it is convenient to express the generators of Γ by half turns as follows.

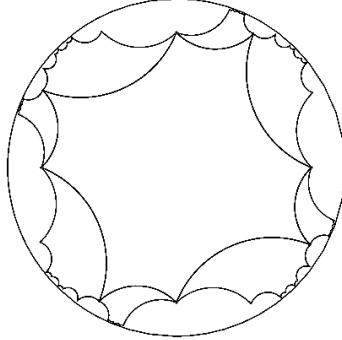


Figure 3: The fundamental octagon with $a = 0.8$, $\alpha = \pi/3$ and the daughter cells.

Let p_k be the mid-point of k -th side, $k = \overline{0,3}$. Since the opposite sides of octagon have the same lengths by construction, the generators are then written as $g_k = H(p_k)$ (see Ref. [16]), where

$$H(p) = \frac{-1}{1 - |p|^2} \begin{pmatrix} 1 + |p|^2 & 2p \\ 2\bar{p} & 1 + |p|^2 \end{pmatrix} \in PSU(1,1). \quad (12)$$

Bar over expression, \bar{p} , means the complex conjugation.

The operation of matrices $H(p)$ consists in the half turn (rotation with angle π) of geodesic segment around the origin $z = 0$ and the half turn around point p .

In our model, all p_k are parametrized by $p_{\pm} \in \mathbb{D} \subset \mathbb{C}$: $p_0 = p_+$, $p_1 = p_-$, $p_2 = p_+e^{i\pi/2}$, and $p_3 = p_-e^{i\pi/2}$. At this time, p_{\pm} are functions of two parameters (a, α) and written down below. In principal, we could first take p_k as “input” and, using then these points, determine the octagon parameters.

To find p_k analytically, set

$$\omega_{\pm} = \frac{be^{i\alpha}(1 - a^2) + ae^{i\pi(1 \mp 1)/4}(1 - b^2)}{1 - a^2b^2}, \quad (13)$$

then one has that

$$p_{\pm} = \frac{\omega_{\pm}}{1 + \sqrt{1 - |\omega_{\pm}|^2}}. \quad (14)$$

The generators g_k can be directly expressed in the terms of auxiliary variables ω_{\pm} . Therefore, $g_k = M(\omega_k)$, where

$$M(\omega) = \frac{-1}{\sqrt{1 - |\omega|^2}} \begin{pmatrix} 1 & \omega \\ \bar{\omega} & 1 \end{pmatrix} \in PSU(1,1), \quad (15)$$

and $\omega_0 = \omega_+$, $\omega_1 = \omega_-$, $\omega_2 = \omega_+e^{i\pi/2}$, $\omega_3 = \omega_-e^{i\pi/2}$.

Now it is possible to build the octagonal lattice with a given fundamental octagon. By the construction, action of the group elements $\gamma \in \Gamma$ on \mathcal{F} produces the daughter cells tiling Poincaré disk,

$$\mathbb{D} = \bigcup_{\gamma \in \Gamma} \gamma \mathcal{F}.$$

The first eight cells $g_k^{\pm 1}\mathcal{F}$ ($k = \overline{0,3}$) are shown in Fig. 3. We see that $g_k^{\pm 1}\mathcal{F} \cap \mathcal{F} \subset \partial\mathcal{F}$ as must be. Of course, this property is preserved for all neighboring cells (not only for fundamental polygon \mathcal{F}).

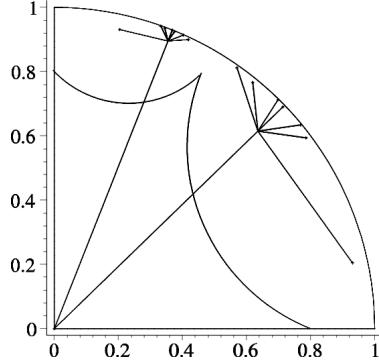


Figure 4: The quarter of fundamental octagon with $a = 0.8$, $\alpha = \pi/3$, and the Cayley tree sites of first and second generations (connected schematically by straight lines but not geodesics).

3 The Directed Random Walk

In this Section, we concern with the directed random (self-avoiding) walk (DRW) on a 8-branching Cayley tree isometrically embedded in Poincaré disk. The tree under consideration is formed by the graph connecting the centers of the neighboring octagons (see Fig. 4) and may be physically interpreted as a model of directed polymers.

We start from the general expression for partition function of DRW in disordered environment on two-dimensional lattice:

$$\mathcal{Z}_N(q) = \sum_{\{z\}} \prod_{t=1}^N p(z_{t-1}|z_t) \exp [qV(z_t, t)], \quad (16)$$

where N is the number of generations; z_t is a point of t -th generation of random trajectory; V is a random potential of environment; p is a transition probability; parameter q can be mapped into an inverse temperature.

In our model, we consider a deterministic motion on graphs of N -th generation, sites of which correspond to a set of numbers $\{i_1, i_2, \dots, i_N\}$, where $1 \leq i_t \leq 8$, $1 \leq t \leq N$. We also assume that

$$z_t = \gamma_{i_1} \gamma_{i_2} \dots \gamma_{i_t}[0], \quad V(z_t, t) = d(0, z_t)/N, \quad (17)$$

$$p(z_{t-1}|z_t) \rightarrow p(i_{t-1}|i_t) = \begin{cases} 1, & |i_t - i_{t-1}| \neq 4 \\ 0, & |i_t - i_{t-1}| = 4 \end{cases}, \quad (18)$$

where $\{\gamma_i | i = \overline{1, 8}\} = \{g_0, g_1, g_2, g_3, g_0^{-1}, g_1^{-1}, g_2^{-1}, g_3^{-1}\}$. By construction, points z_t and z_{t-1} are the centers of neighbouring cells connected by the tree branches. Definition of $p(i_{t-1}|i_t)$ (consistent with definition of generators γ_i , $i = \overline{1, 8}$) excludes a return to the visited sites. Hyperbolic distance $d(z, w)$ on (\mathbb{D}, ds^2) is determined by

$$\cosh d(z, w) = 1 + \frac{2|z - w|^2}{(1 - |z|^2)(1 - |w|^2)}. \quad (19)$$

In particular case, one has

$$d(0, z) = \ln \frac{1 + |z|}{1 - |z|}. \quad (20)$$

In this way, we come to expression for partition function of our model:

$$\mathcal{Z}_N(q) = \sum_{i_1=1}^8 \dots \sum_{i_N=1}^8 \left(\prod_{t=1}^N p(i_{t-1}|i_t) \right) \exp \left[q \frac{1}{N} \sum_{t=1}^N \rho_{i_1, i_2 \dots i_t} \right], \quad (21)$$

where $\rho_{i_1, i_2 \dots i_t} \equiv d(0, \gamma_{i_1} \gamma_{i_2} \dots \gamma_{i_t}[0])$ is the hyperbolic (geodesic) distance between the point of t -th generation and the origin O .

The partition function defined is a sum of Boltzmann weights that depend on an “action integral” linear in the hyperbolic distance from a given root point (the origin O), i.e. the length of trajectory:

$$L(i_1, i_2, \dots, i_N) \equiv \sum_{t=1}^N \rho_{i_1, i_2 \dots i_t}. \quad (22)$$

It is worthy to note that $\mathcal{Z}_N(q)$ is also a function of arbitrary module (a, α) , used for obtaining the generators of the group Γ . To explore the properties of a model described by $\mathcal{Z}_N(q|a, \alpha)$, we shall fix the lattice parameters.

Since the all trajectories are self-avoiding and uniquely determined, the number of graphs $Z_N(0)$ is equal to the number of end-points, $v(N) = 8 \times 7^{N-1}$, and the partition function (21) can be re-written as

$$\mathcal{Z}_N(q) = \sum_{i=1}^{v(N)} e^{qL_i(N)/N}, \quad (23)$$

where $L_i(N)$ is the integrated hyperbolic length of i -th graph of N -th generation.

Thus, the main goal of this work is to evaluate the series (23) dependent on length spectrum $\{L_i(N)\}$. To resolve this problem, we shall first perform numerical investigations to establish basic properties of such a system. Secondly, we shall attempt to calculate the characteristics analytically with the use of statistical approach.

An example of length spectrum obtained numerically is presented in Fig. 5, where the absolute value of the number of events depends on particular choice of the width of shell. We see that the distribution of lengths L_i approaches to the Gaussian distribution,

$$\sim \exp \left(-\frac{(L - \bar{L}_N)^2}{2\sigma_N^2} \right), \quad (24)$$

with parameters defined by formulas

$$\bar{L}_N = \frac{1}{v(N)} \sum_{i=1}^{v(N)} L_i(N), \quad \sigma_N^2 = \frac{1}{v(N)} \sum_{i=1}^{v(N)} (L_i(N) - \bar{L}_N)^2, \quad (25)$$

resulting to $\bar{L}_5 \approx 41.97$, $\sigma_5^2 \approx 44.5657$.

Although the process of such a convergence is slow, it means that the structure generated by the discrete group Γ satisfactory describes a disordered environment that justifies a random walk treatment. Also note that although Fig. 5 was built for the fixed parameters of the lattice, the system reveals the same tendency for any admissible pair (a, α) .

We are giving the further analysis of the model within the framework of the concept of multi-fractality consisting in a scale dependence of critical exponents [8]. To do this, let us introduce the multifractal moments of the order q defined as follows

$$\mathfrak{m}_N^{(q)} = \frac{\mathcal{Z}_N(q)}{\mathcal{Z}_N^q(1)}. \quad (26)$$

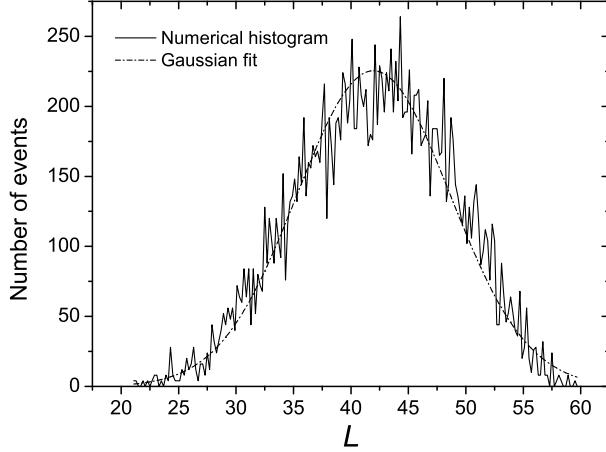


Figure 5: The number of events as the function of the hyperbolic length L ($N = 5$) and the Gaussian fit for lattice with $a = 0.8$, $\alpha = \pi/3$.

Then the scaling exponent, sometimes called as mass spectrum, is

$$\tau_N(q) = \frac{2}{\ln v(N)} \ln m_N^{(q)}, \quad (27)$$

where prefactor 2 corresponds to dimension of underlying geometrical carrier (space) and $v(N)$ coincides with the number of “boxes” tilling structure under consideration.

Exponent $\tau_N(q)$ is self-averaging and therefore it is reasonable to introduce

$$\tau(q) = \lim_{N \rightarrow \infty} \tau_N(q), \quad (28)$$

defining the spectrum of fractal dimensions $D_q = \tau(q)/(1 - q)$.

The result of our numerical calculations (limited to $N \leq 5$) is shown in Fig. 6. We can conclude that the multifractal behavior takes place indeed, that is reflected in non-linear dependence τ on q .

Besides the finding dependence $\tau(q)$, there is another equivalent way to describe the multifractality. It is well known that the set of exponents governing of multifractal moments of type (26) is related to the spectrum of singularities $f(\alpha)$ of fractal measure,[17] called also the spectral function, which is given by the Legendre transform:

$$f(\alpha) = q\alpha + \tau(q), \quad \alpha(q) = -\frac{d\tau(q)}{dq}, \quad (29)$$

where α is known as Lipschitz–Hölder exponent¹. In other words, $f(\alpha)$ denotes the dimension of the subset characterized by the singularity strength α . In the terms of thermodynamics, $\alpha(q)$ coincides with an internal energy e , while $f(\alpha)$ is interpreted as an entropy $s(e)$ [18].

The general properties of $f(\alpha)$ are as follows: it is positive on an interval $[\alpha_{\min}, \alpha_{\max}]$, where

$$\alpha_{\min} = \lim_{q \rightarrow +\infty} \tau(q)/(1 - q), \quad \alpha_{\max} = \lim_{q \rightarrow -\infty} \tau(q)/(1 - q), \quad (30)$$

¹We preserve conventional symbol for this exponent [8]. It will always be clear from context whether α is an angle determining the geometry of octagon or the singularity strength.

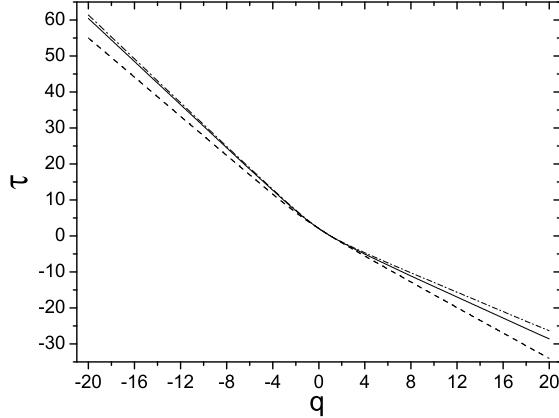


Figure 6: Multifractal behavior of the exponent $\tau(q)$ for $N = 5$. The solid, dashed, and dashed-dotted curves correspond to the lattices with $(a = 0.8, \alpha = \pi/3)$, $(a = 2^{-1/4}, \alpha = \pi/4)$, and $(a = 0.9, \alpha = \pi/8)$, respectively.

and the maximum value of the spectral function gives us the (fractal) dimension of the underlying structure.

Spectral function $f(\alpha)$ obtained for $N = 5$ is given in Fig. 7. In order to compare curves in Fig. 7, we appeal to an information entropy S [8], defined as

$$S = \alpha_S = f(\alpha_S), \quad \left. \frac{df(\alpha)}{d\alpha} \right|_{\alpha_S} = 1. \quad (31)$$

The value of α_S and $f(\alpha_S)$ can be directly calculated at $q = 1$.

We find that $S(a = 0.9, \alpha = \pi/8) \approx 1.783 < S(a = 0.8, \alpha = \pi/3) \approx 1.865 < S(a = 2^{-1/4}, \alpha = \pi/4) \approx 1.95$. This relation has a simple treatment: i) due to a special symmetry, the regular octagon gives us a highest entropy which cannot exceed 2 in our model; ii) deviation from the regular symmetry leads to decreasing the entropy more and more, what says about low probable configurations.

Since the problem of random walk is mathematically the problem of addition of independent random variables, the ultimate behavior of random walkers can be deduced from a central limit theorem (CLT). Thus, another approach, applying CLT, can be also developed.

First, we need to reduce an evolution to the Markov multiplicative process. In order to achieve this, let us use the triangle rule in two-dimensional hyperbolic space:

$$\cosh d(0, z_t) = \cosh d(0, z_{t-1}) \cosh d(z_t, z_{t-1}) - \sinh d(0, z_{t-1}) \sinh d(z_t, z_{t-1}) \cos \theta_{t,t-1}, \quad (32)$$

where $\theta_{t,t-1}$ is angle opposite to side $(0, z_t)$.

Due to identities

$$d(0, \gamma^n[0]) = n d(0, \gamma[0]), \quad d(\tilde{\gamma}[0], \tilde{\gamma}\gamma[0]) = d(0, \gamma[0]), \quad (33)$$

which are valid for matrices (12), (15), we get that $d(z_t, z_{t-1}) = d(0, \gamma_{i_t}[0])$ and

$$\cosh d(0, z_t) = \cosh d(0, z_{t-1}) \cosh \ell_{i_t} - \sinh d(0, z_{t-1}) \sinh \ell_{i_t} \cos \theta_{t,t-1}, \quad (34)$$

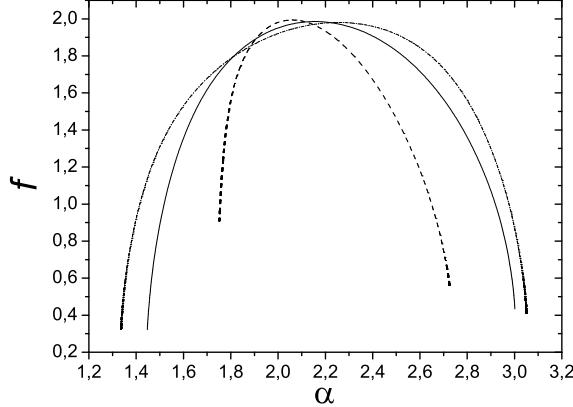


Figure 7: The spectrum of singularities $f(\alpha)$ for $N = 5$. The solid, the dashed, and the dashed-dotted curves correspond to the lattices with $(a = 0.8, \alpha = \pi/3)$, $(a = 2^{-1/4}, \alpha = \pi/4)$, and $(a = 0.9, \alpha = \pi/8)$, respectively.

where, accounting for lattice symmetry, $\ell_i = \delta_i^+ \ell_+ + \delta_i^- \ell_-$,

$$\ell_+ = d(0, \gamma_{2k-1}[0]), \quad \ell_- = d(0, \gamma_{2k}[0]), \quad k = \overline{1, 4}, \quad (35)$$

$$\delta_i^+ = \sum_{k=1}^4 \delta_{i,2k-1}, \quad \delta_i^- = \sum_{k=1}^4 \delta_{i,2k}. \quad (36)$$

For relatively large $d(0, z_t)$ and $d(0, z_{t-1})$, one has

$$d(0, z_t) = d(0, z_{t-1}) + \ln (\cosh \ell_{i_t} - \sinh \ell_{i_t} \cos \theta_{t,t-1}). \quad (37)$$

Now it is useful to introduce

$$\xi_{i,j} = d(0, \gamma_i \gamma_j[0]) - \frac{\ell_i + \ell_j}{2}, \quad i, j = \overline{1, 8}, \quad (38)$$

describing a logarithmic contribution for the second generation.

Then, neglecting fluctuations of $\theta_{t,t-1}$ in higher generations, we obtain an estimation for trajectory length:

$$L(i_1, i_2, \dots, i_N) \simeq N\ell_{i_1} + \sum_{t=2}^N (N+1-t)\xi_{i_{t-1}, i_t} \quad \text{for } p(i_{t-1}|i_t) \neq 0. \quad (39)$$

Note immediately that this formula is exact for $i_1 = i_2 = \dots = i_N$ when $\theta_{t,t-1} = \pi$.

In principal, it is also possible to introduce quantities $\xi_{i_1, i_2, \dots, i_n}^{(n)} \sim d(0, \gamma_{i_1} \gamma_{i_2} \dots \gamma_{i_n}[0])$ playing a role of higher-order correlations and asymmetric by indices in contrast with symmetric matrix $\xi_{i,j} = \xi_{j,i}$. Moreover, matrices $\xi^{(n)}$ are able to account more precisely for values of angles $\theta_{t,t-1}$ or, in another word, fluctuations.

Using Eq. (39), we can easily give the theoretical estimation for the lengths spectrum of N -th generation

$$L_{\min}^{\text{th}}(N) = N \min(\ell_+, \ell_-) + \frac{N(N-1)}{2} \min(\xi_{i,j}), \quad \text{for } p(i|j) \neq 0,$$

$$\begin{aligned}
\bar{L}_N^{\text{th}} &= N \frac{\ell_+ + \ell_-}{2} + \frac{N(N-1)}{2} \bar{\xi}, \quad \bar{\xi} = \frac{1}{56} \sum_{i,j=1}^8 p(i|j) \xi_{i,j}, \\
L_{\max}^{\text{th}}(N) &= N \max(\ell_+, \ell_-) + \frac{N(N-1)}{2} \max(\xi_{i,j}) \\
&= \frac{N(N+1)}{2} \max(\ell_+, \ell_-).
\end{aligned} \tag{40}$$

Substituting (39) in (21), we arrive at the expression for partition function as inhomogeneous Markov multiplicative chain:

$$\mathcal{Z}_N(q) \simeq \sum_{i_1=1}^8 \dots \sum_{i_N=1}^8 \exp(q\ell_{i_1}) \prod_{t=2}^N \mathcal{P}^t(i_{t-1}|i_t), \tag{41}$$

where exponent function plays a role of initial condition, and

$$\mathcal{P}^t(i_{t-1}|i_t) = p(i_{t-1}|i_t) \exp\left[q \frac{N+1-t}{N} \xi_{i_{t-1}, i_t}\right] \tag{42}$$

is transition weight dependent on t . Properties of Eq. (41) are discussed in details in Appendix C.

Thus, we would like to note that the length spectrum $\{L_i(N)\}$ and the corresponding partition function for any N are completely determined by minimal lengths ℓ_{\pm} and matrix $\xi_{i,j}$ in a given approximation.

Now, arranging the lengths of graphs in ascending order $L_1 \leq L_2 \leq \dots \leq L_s$ ($s < v(N)$), we can re-write the partition function (23) as

$$\mathcal{Z}_N(q) = v(N) \sum_{i=1}^s W_i e^{ql_i}, \quad \sum_{i=1}^s W_i = 1, \tag{43}$$

where $l_i = L_i/N$ and W_i denotes the weight of the orbit of length L_i .

Hereafter, coefficients W_i are assumed to be randomly distributed. It means that the value of W_i is independent of W_k for $i \neq k$, allowing the application of the random walk model.

Next, assuming that the CLT is valid at $N \rightarrow \infty$ for our system, let us model a distribution of the lengths l_i of N -th generation graphs by the Gaussian distribution:

$$W_N(l) = A_N \exp\left(-\frac{(l - N\bar{l})^2}{2Ns^2}\right), \tag{44}$$

where $N\bar{l} = \bar{L}_N/N$, $Ns^2 = \sigma_N^2/N^2$ when $N \rightarrow \infty$; A_N is a normalization constant. We can theoretically evaluate \bar{l} and s^2 by the use of Eqs. (39), (40).

We have already seen in Fig. 5 that the Gaussian approximation is in good agreement with numerical data. So, it is believed that an application of Gaussian distribution may simplify significantly the calculations of the series over group Γ .

Applying the approximate distribution (44), we can replace the sums (43) with integrals at large N :

$$\mathcal{Z}_N(q) \approx v(N) \int_{\ell_{\min}(N)}^{\ell_{\max}(N)} W_N(l) e^{ql} dl, \quad \int_{\ell_{\min}(N)}^{\ell_{\max}(N)} W_N(l) dl = 1, \tag{45}$$

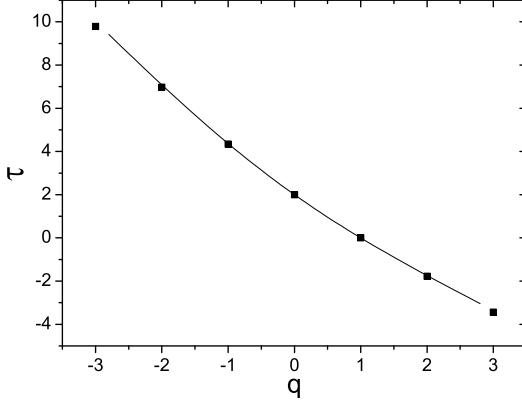


Figure 8: Multifractal behavior of the exponent $\tau(q)$ for $N = 5$ and $(a = 0.8, \alpha = \pi/3)$. The points and the curve are obtained on the base of partition functions (23), (46), respectively.

where $\ell_{\min}(N) = L_{\min}(N)/N$ and $\ell_{\max}(N) = L_{\max}(N)/N$ are introduced in order to cut long Gaussian tails.

After simple calculations we obtain that

$$\mathcal{Z}_N(q) \approx \frac{8}{7} C_N(q) \exp N \left(\frac{1}{2} q^2 s^2 + q \bar{l} + \ln 7 \right), \quad (46)$$

where coefficient

$$C_N(q) = \frac{\operatorname{erf} \left(\frac{\ell_{\max}(N) - N \bar{l} - q N s^2}{\sqrt{2 N s^2}} \right) + \operatorname{erf} \left(\frac{N \bar{l} + q N s^2 - \ell_{\min}(N)}{\sqrt{2 N s^2}} \right)}{\operatorname{erf} \left(\frac{\ell_{\max}(N) - N \bar{l}}{\sqrt{2 N s^2}} \right) + \operatorname{erf} \left(\frac{N \bar{l} - \ell_{\min}(N)}{\sqrt{2 N s^2}} \right)} \quad (47)$$

is responsible for surface effects which are not essential in infinite Bethe lattice;

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \quad (48)$$

Comparison of the values of $\tau(q)$ obtained numerically and calculated on the base of Eq. (46) is presented in Fig 8. We can conclude that a theoretical formula works in the limited interval of q (small $|q|$) and satisfactorily describes exponent $\tau(q)$ for relatively small $1 < N < \infty$. However, we can note that there is a possibility to improve theoretical result by inclusion of higher-order correlations $\xi^{(n)}$, that leads to the processes beyond Markov theory.

4 Discussion

Generally, we have concerned here with calculation of the series over Fuchsian group as group of symmetries of hyperbolic octagonal lattice with one independent complex module. To develop an analytical approach, we have considered the model of directed random walk on 8-branching Cayley tree as dual lattice for a given octagonal lattice. Then, the truncated series over symmetry group was treated as the partition function with Boltzmann weights for ensemble of lengths of tree graphs.

Each of these lengths was defined by the sum of hyperbolic distances between the tree sites visited by directed walker and the origin.

As the result, we have analytically derived the spectrum of lengths (39) leading to the partition function (41) in the form of the Markov multiplicative chain allowing us to apply the central limit theorem. It says about randomizing environment, where spatial evolution is developed, and applicability of a random walk method in our calculations. Moreover, we have explored the model within the multifractality concept giving us a spectrum of scaling exponents.

In principal, the investigations performed might be mapped into the problem of calculation of the Selberg trace [19], intensively used for quantization of systems on the Riemann surfaces of genus two and requiring the knowledge of geodesics length spectrum (see Ref. [15], for instance). Also, in order to obtain the field observables on Riemann surface of the fixed genus and geometry, we can “symmetrize” these quantities under the action of corresponding group, that often coincides with the Dirichlet or Poincaré series computation. We only note a slow convergence of these series, that requires to use the accurate approximations which we have touched in this paper.

Appendix A. Curved Space vs Potential Field

Let us consider a particle in an attraction potential field $U(x, y) > 0$ in Euclidean space. The Lagrangean function and the energy are

$$L = \frac{1}{2} \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right] + \frac{1}{2} U(x, y), \quad (49)$$

$$E = \frac{1}{2} \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right] - \frac{1}{2} U(x, y). \quad (50)$$

Let potential obey the Liouville equation,

$$\Delta_{\mathbb{E}} \ln U = -2KU, \quad \Delta_{\mathbb{E}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad (51)$$

with $K = -1$ (negative Gaussian curvature); $\Delta_{\mathbb{E}}$ is the Laplace operator in Euclidean two-dimensional space.

Limiting by the case of central symmetry, when $U = U(r)$, $r = \sqrt{x^2 + y^2}$, equation (51) takes the form:

$$U'' + \frac{1}{r} U' - \frac{(U')^2}{U} = 2U^2, \quad (52)$$

where $U' = dU(r)/dr$.

Solution to Eq. (52) is

$$U(r) = \frac{A^2}{r^2 \sinh^2(A \ln r + C)}, \quad (53)$$

where A, C are arbitrary constants.

Following the consequence of Maupertuis variational principle, at the fixed value of the energy E , the truncated action function s and the time t are determined from differential relations:

$$ds^2 = (2E + U)(dr^2 + r^2 d\varphi^2), \quad (54)$$

$$dt^2 = \frac{dr^2 + r^2 d\varphi^2}{2E + U}, \quad (55)$$

where $\varphi = \arctan(y/x)$.

Such a description of dynamical system is based on the equivalence of the trajectories within the usual Lagrangean formalism and the geodesics in space with metric (54).

In the case $E = 0$, we arrive at the metric:

$$ds^2 = U(r)(dr^2 + r^2 d\varphi^2) \quad (56)$$

with Gaussian curvature $K = -1$.

Thus we have done transition from the problem of the particle in the potential field in Euclidean space to the problem of the free particle in the curved Lobachevsky space. We find that the metric (2) is realized when $A = 1$, $C = 0$.

Since the metric (56) after replacement $r \rightarrow \exp(-C/A)r$, $\varphi \rightarrow \varphi/A$ gives us (2), these metrics are isometric.

Coming back to Eq. (55), relation between t (time in Euclidean model) and s (proper time in Poincaré model now) is derived from expression:

$$\begin{aligned} t &= \int \frac{1}{2}(1-x^2-y^2)\sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2} ds \\ &= \frac{R^2\sqrt{1+R^2}\sinh s}{\sqrt{1+R^2}\cosh s + R} - 2R^3 \arctan \left[\left(\sqrt{1+R^2} - R \right) \tanh \frac{s}{2} \right]. \end{aligned} \quad (57)$$

Then one finds that trajectories for the problem (49)-(50) can be defined in parametric form as $\{t, x^i(t)\} = \{t(s), x^i(s)|s \in \mathbb{R}\}$, and

$$\frac{dx^i}{dt} = \frac{dx^i}{ds} \left(\frac{dt}{ds} \right)^{-1}. \quad (58)$$

Appendix B.

In order to describe completely the geometry of fundamental domain \mathcal{F} in our model, it is necessary to find seven quantities,

$$R_{\pm}, \phi_{\pm}, b, \beta, \gamma,$$

as functions of a and α (see Fig. 1). Parameters R_{\pm} , ϕ_{\pm} determine the geodesics forming edges of octagon, b (or $be^{i(\alpha+k\pi/2)}$, $k = \overline{0,3}$) determines the location of four vertices of octagon, β and γ are the angles by vertices $ae^{ik\pi/2}$ and $be^{i(\alpha+k\pi/2)}$, respectively. Here we write down seven equations giving us the values of these parameters in terms of a and α .

The first equation results from the Gauss–Bonnet theorem for surfaces of genus two:

$$\beta + \gamma = \frac{\pi}{2}. \quad (59)$$

Next six equations are derived by using the equations of geodesics in the following (complex) form:

$$\left| z - \sqrt{1+R_{\pm}^2} e^{i(\phi_{\pm}+k\pi/2)} \right|^2 = R_{\pm}^2, \quad k = \overline{0,3}, \quad z \in \mathbb{D}. \quad (60)$$

Due to the special symmetry of octagon, it is quite enough to formulate the necessary relations at two vertices: $z = a$ and $z = be^{i\alpha}$. These relations follow from the requirement of intersection of two

geodesics, labeled as \pm , and are of the form:

$$1 + a^2 - 2a\sqrt{1 + R_+^2} \cos \phi_+ = 0, \quad (61)$$

$$1 + a^2 - 2a\sqrt{1 + R_-^2} \sin \phi_- = 0, \quad (62)$$

$$\begin{aligned} & a^2 - a \left(\sqrt{1 + R_+^2} \cos \phi_+ + \sqrt{1 + R_-^2} \sin \phi_- \right) \\ & + \sqrt{1 + R_+^2} \sqrt{1 + R_-^2} \sin(\phi_- - \phi_+) - R_+ R_- \cos \beta = 0, \end{aligned} \quad (63)$$

and

$$1 + b^2 - 2b\sqrt{1 + R_+^2} \cos(\alpha - \phi_+) = 0, \quad (64)$$

$$1 + b^2 - 2b\sqrt{1 + R_-^2} \cos(\phi_- - \alpha) = 0, \quad (65)$$

$$\begin{aligned} & b^2 - b \left(\sqrt{1 + R_+^2} \cos(\alpha - \phi_+) + \sqrt{1 + R_-^2} \cos(\phi_- - \alpha) \right) \\ & + \sqrt{1 + R_+^2} \sqrt{1 + R_-^2} \sin(\phi_- - \phi_+) - R_+ R_- \cos \gamma = 0, \end{aligned} \quad (66)$$

where $\phi_+ < \alpha < \phi_-$ by construction.

As it has been noted in main text of the paper, solution to these equations is expressions (4)–(8) derived after simple but cumbersome computations.

Appendix C.

In this Section, we would like to discuss some properties of the kernel of multiplicative chain (41) defined at small $|q|$:

$$K_{i_1, i_N}(q, N) \equiv \sum_{i_2=1}^8 \dots \sum_{i_{N-1}=1}^8 \prod_{t=2}^N \mathcal{P}^t(i_{t-1}|i_t). \quad (67)$$

First, introducing the vector

$$K_{i_1}(q, N) \equiv \sum_{i_N=1}^8 K_{i_1, i_N}(q, N), \quad (68)$$

it is worthy to note that this expression allows one the following representation:

$$K_i(q, N) = K_+(q, N)\delta_i^+ + K_-(q, N)\delta_i^-, \quad (69)$$

which is obtained on the base of semi analytical observations and reflects the symmetry of the lattice in our model; δ_i^\pm is defined by Eq. (36). Due to this expansion, Eq. (41) is re-written as

$$\mathcal{Z}_N(q) \simeq 4 \exp(q\ell_+) K_+(q, N) + 4 \exp(q\ell_-) K_-(q, N). \quad (70)$$

Next, it is useful to define the probability of transition from initial state i to final state f for N steps:

$$P_{i,f}(q, N) \equiv \frac{K_{i,f}(q, N)}{K_i(q, N)}, \quad \sum_{f=1}^8 P_{i,f}(q, N) = 1. \quad (71)$$

In principal, using $P_{i,f}(q, N)$, one can determine a more probable final state f for a given q , N , and initial state i . We can also verify that

$$P_{i,f}(0, N \rightarrow \infty) \rightarrow \frac{1}{8}. \quad (72)$$

It means that the chain (41) is ergodic at $q = 0$. The same conclusion is obtained by investigation of the mean value \bar{l} (the mean step of graphs) which is introduced in Eq. (44) and equal to

$$\bar{l} \equiv \lim_{N \rightarrow \infty} \frac{\bar{L}}{N^2} \rightarrow \frac{1}{2}\bar{\xi}. \quad (73)$$

This result is also independent on initial condition.

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